

# ALMOST PERIODIC GENERALIZED FUNCTIONS

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**ABSTRACT.** The aim of this paper is to introduce and to study an algebra of almost periodic generalized functions containing the classical Bohr almost periodic functions as well as almost periodic Schwartz distributions.

## 1. INTRODUCTION

The theory of uniformly almost periodic functions was introduced and studied by H. Bohr, since then, many authors contributed to the development of this theory. There exist three equivalent definitions of uniformly almost periodic functions, the first definition of H. Bohr, S. Bochner's definition and the definition based on the approximation property, see [1]. S. Bochner's definition is more suitable for extension to Schwartz distributions. L. Schwartz in [6] introduced the basic elements of almost periodic distributions.

The new generalized functions of [2], [3], give a solution to the problem of multiplication of distributions, these generalized functions are currently the subject of many scientific works, see [4] and [5].

The aim of this work is to introduce and to study an algebra of almost periodic generalized functions containing the classical Bohr almost periodic functions as well almost periodic Schwartz distributions.

## 2. ALMOST PERIODIC FUNCTIONS AND DISTRIBUTIONS

We consider functions and distributions defined on the whole one dimensional space  $\mathbb{R}$ . Recall  $\mathcal{C}_b$  the space of bounded and continuous complex valued functions on  $\mathbb{R}$  endowed with the norm  $\| \cdot \|_\infty$  of uniform convergence on  $\mathbb{R}$ ,  $(\mathcal{C}_b, \| \cdot \|_\infty)$  is a Banach algebra.

**Definition 1.** (*S. Bochner*) A complex valued function  $f$  defined and continuous on  $\mathbb{R}$  is called almost periodic, if for any sequence of real numbers  $(h_n)_n$  one can extract a subsequence  $(h_{n_k})_k$  such that

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$(f(\cdot + h_{n_k}))_k$  converges in  $(\mathcal{C}_b, \|\cdot\|_\infty)$ . Denoted by  $\mathcal{C}_{ap}$  the space of almost periodic functions.

To recall Schwartz almost periodic distributions, we need some function spaces, see [6]. Let  $p \in [1, +\infty]$ , the space

$$\mathcal{D}_{L^p} := \{\varphi \in \mathcal{C}^\infty : \varphi^{(j)} \in L^p, \forall j \in \mathbb{Z}_+\}$$

endowed with the topology defined by the countable family of norms

$$|\varphi|_{k,p} := \sum_{j \leq k} \|\varphi^{(j)}\|_{L^p}, \quad k \in \mathbb{Z}_+,$$

is a differential Frechet subalgebra of  $\mathcal{C}^\infty$ . The topological dual of  $\mathcal{D}_{L^1}$ , denoted by  $\mathcal{D}'_{L^\infty}$ , is called the space of bounded distributions.

Let  $h \in \mathbb{R}$  and  $T \in \mathcal{D}'$ , the translate of  $T$  by  $h$ , denoted by  $\tau_h T$ , is defined as :

$$\langle \tau_h T, \varphi \rangle = \langle T, \tau_{-h} \varphi \rangle, \quad \varphi \in \mathcal{D},$$

where  $\tau_{-h} \varphi(x) = \varphi(x + h)$ .

The definition and characterizations of an almost periodic distribution are summarized in the following results.

**Theorem 1.** *For any bounded distribution  $T \in \mathcal{D}'_{L^\infty}$ , the following statements are equivalent :*

- i) *The set  $\{\tau_h T, h \in \mathbb{R}\}$  is relatively compact in  $\mathcal{D}'_{L^\infty}$ .*
- ii)  *$T * \varphi \in \mathcal{C}_{ap}, \forall \varphi \in \mathcal{D}$ .*
- iii)  *$\exists (f_j)_{j \leq k} \subset \mathcal{C}_{ap}, T = \sum_{j \leq k} f_j^{(j)}$ .*

$T \in \mathcal{D}'_{L^\infty}$  is said almost periodic if it satisfies any (hence every) of the above conditions.

**Definition 2.** *The space of almost periodic distributions is denoted by  $\mathcal{B}'_{ap}$ .*

Let recall the space of regular almost periodic functions.

**Definition 3.** *The space of almost periodic infinitely differentiable functions on  $\mathbb{R}$  is defined and denoted by*

$$\mathcal{B}_{ap} = \{\varphi \in \mathcal{D}_{L^\infty} : \varphi^{(j)} \in \mathcal{C}_{ap}, \forall j \in \mathbb{Z}_+\}.$$

Some, easy to prove, properties of  $\mathcal{B}_{ap}$  are given in the following assertions.

**Proposition 1.** *We have*

- i)  $\mathcal{B}_{ap}$  *is a closed differential subalgebra of  $\mathcal{D}_{L^\infty}$ .*
- ii) *If  $T \in \mathcal{B}'_{ap}$  and  $\varphi \in \mathcal{B}_{ap}$ , then  $\varphi T \in \mathcal{B}'_{ap}$ .*
- iii)  $\mathcal{B}_{ap} * L^1 \subset \mathcal{B}_{ap}$ .
- iv)  $\mathcal{B}_{ap} = \mathcal{D}_{L^\infty} \cap \mathcal{C}_{ap}$ .

As a consequence of (iv), we have the following result.

**Corollary 1.** *If  $v \in \mathcal{D}_{L^\infty}$  and  $v * \varphi \in \mathcal{C}_{ap}$ ,  $\forall \varphi \in \mathcal{D}$ , then  $v \in \mathcal{B}_{ap}$ .*

**Remark 1.** *It is important to mention that  $\mathcal{B}_{ap} \subsetneq \mathcal{C}^\infty \cap \mathcal{C}_{ap}$ .*

### 3. ALMOST PERIODIC GENERALIZED FUNCTIONS

Let  $I = ]0, 1]$  and

$$\mathcal{M}_{L^\infty} = \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^\infty})^I, \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O(\varepsilon^{-m}), \varepsilon \rightarrow 0 \right\}$$

$$\mathcal{N}_{L^\infty} = \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^\infty})^I, \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O(\varepsilon^m), \varepsilon \rightarrow 0 \right\}$$

**Definition 4.** *The algebra of bounded generalized functions, denoted by  $\mathcal{G}_{L^\infty}$ , is defined by the quotient*

$$\mathcal{G}_{L^\infty} = \frac{\mathcal{M}_{L^\infty}}{\mathcal{N}_{L^\infty}}$$

Define

(3.1)

$$\mathcal{M}_{ap} = \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{ap})^I, \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O(\varepsilon^{-m}), \varepsilon \rightarrow 0 \right\}$$

$$\mathcal{N}_{ap} = \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{ap})^I, \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O(\varepsilon^m), \varepsilon \rightarrow 0 \right\}$$

The properties of  $\mathcal{M}_{ap}$  and  $\mathcal{N}_{ap}$  are summarized in the following proposition.

**Proposition 2.** *i) The space  $\mathcal{M}_{ap}$  is a subalgebra of  $(\mathcal{B}_{ap})^I$ .*

*ii) the space  $\mathcal{N}_{ap}$  is an ideal of  $\mathcal{M}_{ap}$ .*

*Proof.* i) It follows from the fact that  $\mathcal{B}_{ap}$  is an differential algebra.

ii) Let  $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{ap}$  and  $(v_\varepsilon)_\varepsilon \in \mathcal{M}_{ap}$ , we have

$$\forall k \in \mathbb{Z}_+, \exists m' \in \mathbb{Z}_+, \exists c_1 > 0, \exists \varepsilon_0 \in I, \forall \varepsilon < \varepsilon_0, |v_\varepsilon|_{k,\infty} < c_1 \varepsilon^{-m'}.$$

Take  $m \in \mathbb{Z}_+$ , then for  $m'' = m + m'$ ,  $\exists c_2 > 0$  such that  $|u_\varepsilon|_{k,\infty} < c_2 \varepsilon^{m''}$ . Since the family of the norms  $|u_\varepsilon|_{k,\infty}$  is compatible with the algebraic structure of  $\mathcal{D}_{L^\infty}$ , then  $\forall k \in \mathbb{Z}_+$ ,  $\exists c_k > 0$  such that

$$|u_\varepsilon v_\varepsilon|_{k,\infty} \leq c_k |u_\varepsilon|_{k,\infty} |v_\varepsilon|_{k,\infty},$$

consequently

$$|u_\varepsilon v_\varepsilon|_{k,\infty} < c_k c_2 \varepsilon^{m''} c_1 \varepsilon^{-m'} \leq C \varepsilon^m, \text{ where } C = c_1 c_2 c_k.$$

Hence  $(u_\varepsilon v_\varepsilon)_\varepsilon \in \mathcal{N}_{ap}$ . □

The following definition introduces the algebra of almost periodic generalized functions.

**Definition 5.** *The algebra of almost periodic generalized functions is the quotient algebra*

$$\mathcal{G}_{ap} = \frac{\mathcal{M}_{ap}}{\mathcal{N}_{ap}}$$

We have a characterization of elements of  $\mathcal{G}_{ap}$  similar to the result (ii) of theorem (1) for almost periodic distributions.

**Theorem 2.** *Let  $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{L^\infty}$ , the following assertions are equivalent :*

- i)  $u$  is almost periodic.
- ii)  $u_\varepsilon * \varphi \in \mathcal{B}_{ap}$ ,  $\forall \varepsilon \in I, \forall \varphi \in \mathcal{D}$ .

*Proof.* i)  $\implies$  ii) If  $u \in \mathcal{G}_{ap}$ , so for every  $\varepsilon \in I$  we have  $u_\varepsilon \in \mathcal{B}_{ap}$ , then  $u_\varepsilon * \varphi \in \mathcal{B}_{ap}$ ,  $\forall \varepsilon \in I, \forall \varphi \in \mathcal{D}$ .

ii)  $\implies$  i) Let  $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{L^\infty}$  and  $u_\varepsilon * \varphi \in \mathcal{B}_{ap}$ ,  $\forall \varepsilon \in I, \forall \varphi \in \mathcal{D}$ , therefor  $u_\varepsilon \in \mathcal{B}_{ap}$  follows from theorem (1) (ii); it suffices to show that

$$\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O(\varepsilon^{-m}), \varepsilon \longrightarrow 0,$$

which follows from the fact that  $u \in \mathcal{G}_{L^\infty}$ .  $\square$

**Remark 2.** *The characterization (ii) does not depend on representatives.*

**Definition 6.** *Denote by  $\Sigma$  the subset of functions  $\rho \in \mathcal{S}$  satisfying*

$$\int \rho(x) dx = 1 \text{ and } \int x^k \rho(x) dx = 0, \forall k = 1, 2, \dots$$

Set  $\rho_\varepsilon(\cdot) = \frac{1}{\varepsilon} \rho(\frac{\cdot}{\varepsilon})$ ,  $\varepsilon > 0$ .

**Proposition 3.** *Let  $\rho \in \Sigma$ , the map*

$$\begin{aligned} i_{ap} : \mathcal{B}'_{ap} &\longrightarrow \mathcal{G}_{ap} \\ u &\longrightarrow (u * \rho_\varepsilon)_\varepsilon + \mathcal{N}_{ap}, \end{aligned}$$

*is a linear embedding which commutes with derivatives.*

*Proof.* Let  $u \in \mathcal{B}'_{ap}$ , by characterization of almost periodic distributions we have  $u = \sum_{\beta \leq m} f_\beta^{(\beta)}$ , where  $f_\beta \in \mathcal{C}_{ap}$ , so  $\forall \alpha \in \mathbb{Z}$ ,

$$|(u^{(\alpha)} * \rho_\varepsilon)(x)| \leq \sum_{\beta \leq m} \frac{1}{\varepsilon^{\alpha+\beta}} \int_{\mathbb{R}} |f_\beta(x - \varepsilon y) \rho^{(\alpha+\beta)}(y)| dy,$$

consequently, there exists  $c > 0$  such that

$$\sup_{x \in \mathbb{R}} |(u^{(\alpha)} * \rho_\varepsilon)(x)| \leq \sum_{\beta \leq m} \frac{1}{\varepsilon^{\alpha+\beta}} \|f_\beta\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\rho^{(\alpha+\beta)}(y)| dy \leq \frac{c}{\varepsilon^{\alpha+m}},$$

i.e.

$$|u * \rho_\varepsilon|_{m', \infty} = \sum_{\alpha \leq m'} \sup_{x \in \mathbb{R}} |(u^{(\alpha)} * \rho_\varepsilon)(x)| \leq \frac{c'}{\varepsilon^{m+m'}}, \quad c' = \sum_{\alpha \leq m'} \frac{c}{\varepsilon^\alpha},$$

this shows that  $(u * \rho_\varepsilon)_\varepsilon \in \mathcal{M}_{ap}$ . Let  $(u * \rho_\varepsilon)_\varepsilon \in \mathcal{N}_{ap}$ , then  $\lim_{\varepsilon \rightarrow 0} u * \rho_\varepsilon = 0$  in  $\mathcal{D}'_{L^\infty}$ , but  $\lim_{\varepsilon \rightarrow 0} u * \rho_\varepsilon = u$  in  $\mathcal{D}'_{L^\infty}$ , this shows that  $i_{ap}$  is an embedding. Finally we note that  $i_{ap}$  is linear, this results from the fact that the convolution is linear and that  $i_{ap}(w^{(j)}) = (w^{(j)} * \rho_\varepsilon)_\varepsilon = (w * \rho_\varepsilon)_\varepsilon^{(j)} = (i_{ap}(w))^{(j)}$ .  $\square$

The space  $\mathcal{B}_{ap}$  is embedded into  $\mathcal{G}_{ap}$  canonically, i.e.

$$\begin{array}{ccc} \sigma_{ap} : \mathcal{B}_{ap} & \longrightarrow & \mathcal{G}_{ap} \\ f & \longrightarrow & [(f)_\varepsilon] = (f)_\varepsilon + \mathcal{N}_{ap} \end{array}$$

There is two ways to embed  $f \in \mathcal{B}_{ap}$  into  $\mathcal{G}_{ap}$ . Actually we have the same result.

**Proposition 4.** *The following diagram*

$$\begin{array}{ccc} \mathcal{B}_{ap} & \longrightarrow & \mathcal{B}'_{ap} \\ & \searrow \sigma_{ap} & \downarrow i_{ap} \\ & & \mathcal{G}_{ap} \end{array}$$

*is commutative.*

*Proof.* Let  $f \in \mathcal{B}_{ap}$ , we prove that  $(f * \rho_\varepsilon - f)_\varepsilon \in \mathcal{N}_{ap}$ . By Taylor's formula and the fact that  $\rho \in \Sigma$ , we obtain

$$\|f * \rho_\varepsilon - f\|_{L^\infty} \leq \varepsilon^m \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} \left| \frac{(-y)^m}{m!} f^{(m)}(x - \theta(x)\varepsilon y) \rho(y) dy \right|,$$

then  $\exists C_m > 0$ , such that

$$\|f * \rho_\varepsilon - f\|_{L^\infty} \leq \varepsilon^m C_m \|f^{(m)}\|_{L^\infty} \|y^m \rho\|_{L^1}.$$

The same result can be obtained for all the derivatives of  $f$ . Hence  $(f * \rho_\varepsilon - f)_\varepsilon \in \mathcal{N}_{ap}$ .  $\square$

The Colombeau algebra of tempered generalized functions on  $\mathbb{C}$  is denoted  $\mathcal{G}_{\mathcal{T}}(\mathbb{C})$ , for more details on  $\mathcal{G}_{\mathcal{T}}(\mathbb{C})$  see [2] or [4].

**Proposition 5.** *Let  $u \in \mathcal{G}_{ap}$  and  $F \in \mathcal{G}_{\mathcal{T}}(\mathbb{C})$ , then*

$$F \circ u = [(F \circ u_\varepsilon)_\varepsilon]$$

*is a well defined element of  $\mathcal{G}_{ap}$ .*

*Proof.* It follows from the classical case of composition, in context of Colombeau algebra, we have  $F \circ u_\varepsilon \in \mathcal{B}_{ap}$  in view of the classical results of composition and convolution.  $\square$

We recall a characterization of integrable distributions.

**Definition 7.** A distribution  $v \in \mathcal{D}'$  is said an integrable distribution, denoted  $v \in \mathcal{D}'_{L^1}$ , if and only if  $v = \sum_{i \leq l} f_i^{(i)}$ , where  $f_i \in L^1$ .

**Proposition 6.** If  $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{ap}$  and  $v \in \mathcal{D}'_{L^1}$ , then the convolution  $u * v$  defined by

$$(u * v)(x) = \left( \int_{\mathbb{R}} u_\varepsilon(x - y) v(y) dy \right)_\varepsilon + \mathcal{N}[\mathbb{C}]$$

is a well defined almost periodic generalized function.

*Proof.* Let  $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{ap}$  be a representative of  $u$ , then

$$\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, \exists C > 0, \exists \varepsilon_0 \in I, \forall \varepsilon \leq \varepsilon_0, |u_\varepsilon|_{k, \infty} < C\varepsilon^{-m},$$

since  $v \in \mathcal{D}'_{L^1}$  then  $v = \sum_{i \leq l} f_i^{(i)}$ , where  $f_i \in L^1$ . For each  $\varepsilon \in I$ ,  $u_\varepsilon * v$  is an almost periodic infinitely differentiable function. By Young inequality there exists  $C > 0$  such that

$$\left\| (u_\varepsilon * v)^{(j)} \right\|_{L^\infty} \leq C \sum_{i \leq l} \|f_i\|_{L^1} \|u_\varepsilon^{(i+j)}\|_{L^\infty},$$

consequently

$$|u_\varepsilon * v|_{k, \infty} = O(\varepsilon^{-m}), \varepsilon \rightarrow 0,$$

this shows that  $(u_\varepsilon * v)_\varepsilon \in \mathcal{M}_{ap}$ . Suppose that  $(w_\varepsilon)_\varepsilon \in \mathcal{M}_{ap}$  is another representative of  $u$ , then there exists  $C > 0$  such that

$$\begin{aligned} \|(u_\varepsilon * v - w_\varepsilon * v)\|_{L^\infty} &\leq \sum_{i \leq l} \sup_{\mathbb{R}} \int_{\mathbb{R}} \left| (u_\varepsilon - w_\varepsilon)^{(i)}(x - y) \right| |f_i(y)| dy \\ &\leq C \sum_{i \leq l} \|f_i\|_{L^1} \left\| (u_\varepsilon - w_\varepsilon)^{(i)} \right\|_{L^\infty}, \end{aligned}$$

as  $(u_\varepsilon - w_\varepsilon)_\varepsilon \in \mathcal{N}_{ap}$ , so  $\forall m \in \mathbb{Z}_+$ ,

$$|(u_\varepsilon * v - w_\varepsilon * v)(x)| = O(\varepsilon^m), \varepsilon \rightarrow 0.$$

We obtain the same result for  $(u_\varepsilon * v - w_\varepsilon * v)_\varepsilon^{(j)}$ . Hence  $(u_\varepsilon * v - w_\varepsilon * v)_\varepsilon \in \mathcal{N}_{ap}$ .  $\square$

If  $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{ap}$ , taking the integral of each element  $u_\varepsilon$  on a compact, we obtain an element of  $\mathbb{C}^I$ .

**Definition 8.** Let  $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{ap}$  and  $x_0 \in \mathbb{R}$ , define the primitive of  $u$  by

$$U(x) = \left( \int_{x_0}^x u_\varepsilon(t) dt \right)_\varepsilon + \mathcal{N}[\mathbb{C}].$$

We give a generalized version of the classical Bohl-Bohr theorem.

**Proposition 7.** *The primitive of an almost periodic generalized function is almost periodic if and only if it is bounded generalized function.*

*Proof.*  $(\implies)$  : It follows from the fact that  $\mathcal{G}_{ap} \subset \mathcal{G}_{L^\infty}$ .  $(\impliedby)$  : Let  $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{ap}$  and let  $U = [(U_\varepsilon)_\varepsilon]$  be its primitive, since for each  $\varepsilon \in I$ ,  $u_\varepsilon \in \mathcal{B}_{ap}$  and  $U_\varepsilon = \int_{x_0}^x u_\varepsilon(t) dt \in \mathcal{D}_{L^\infty}$  then by the classical result

of Bohl-Bohr and  $\mathcal{B}_{ap}$ , for every  $\varepsilon \in I$  we have  $\int_{x_0}^x u_\varepsilon(t) dt \in \mathcal{B}_{ap}$ . We shows that  $(U_\varepsilon)_\varepsilon \in \mathcal{M}_{ap}$ , i.e.

$$\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |U_\varepsilon|_{k,\infty} = \sum_{j \leq k} \sup_{x \in \mathbb{R}} |U_\varepsilon^{(j)}(x)| = O(\varepsilon^{-m}), \varepsilon \longrightarrow 0.$$

If  $j = 0$ , since  $(U_\varepsilon)_\varepsilon \in \mathcal{M}_{L^\infty}$ . By hypothesis, we have

$$\sup_{x \in \mathbb{R}} |U_\varepsilon(x)| = O(\varepsilon^{-m}), \varepsilon \longrightarrow 0,$$

which shows that  $(U_\varepsilon)_\varepsilon \in \mathcal{M}_{ap}$ . If  $j \geq 1$ , we have  $|U_\varepsilon^{(j)}(x)| = |u_\varepsilon^{(j-1)}(x)|$ , which gives

$$\sum_{j \leq k} \sup_{x \in \mathbb{R}} |U_\varepsilon^{(j)}(x)| \leq \sum_{j \leq k} \sup_{x \in \mathbb{R}} |u_\varepsilon^{(j-1)}(x)|,$$

consequently

$$|U_\varepsilon|_{k,\infty} = O(\varepsilon^{-m}), \varepsilon \longrightarrow 0,$$

i.e.  $(U_\varepsilon)_\varepsilon \in \mathcal{M}_{ap}$ . □

As in the classical theory, we introduce the notion of mean value within the algebra  $\mathcal{G}_{ap}$ .

**Definition 9.** Let  $u \in \mathcal{G}_{ap}$ , the generalized mean value of  $u$ , denoted by  $M_g(u)$ , is defined by

$$M_g(u) = \left( \lim_{X \rightarrow +\infty} \frac{1}{X} \int_0^X u_\varepsilon(x) dx \right)_\varepsilon + \mathcal{N}[\mathbb{C}],$$

where  $(u_\varepsilon)_\varepsilon$  is a representative of  $u$ .

The definition of  $M_g(u)$  is correct and does not depend on representatives.

**Proposition 8.** *i) If  $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{ap}$ , then  $M_g(u) \in \tilde{\mathbb{C}}$ .*

*ii) If  $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{ap}$ , then  $M_g(u) = 0$  in  $\tilde{\mathbb{C}}$ .*

*Proof.* i) Let  $\varepsilon \in I$ , we have

$$\left| \lim_{X \rightarrow +\infty} \frac{1}{X} \int_0^X u_\varepsilon(x) dx \right| \leq \sup_{x \in \mathbb{R}} |u_\varepsilon(x)|,$$

as  $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{ap}$ , so  $\exists m \in \mathbb{Z}_+$ ,  $\sup_{x \in \mathbb{R}} |u_\varepsilon(x)| = O(\varepsilon^{-m})$ ,  $\varepsilon \rightarrow 0$ , hence

$$\left( \lim_{X \rightarrow +\infty} \frac{1}{X} \int_0^X u_\varepsilon(x) dx \right)_\varepsilon \in \mathcal{E}_M[\mathbb{C}].$$

ii) If  $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{ap}$ , i.e.

$$\left| \lim_{X \rightarrow +\infty} \frac{1}{X} \int_0^X u_\varepsilon(x) dx \right| = O(\varepsilon^m), \varepsilon \rightarrow 0, \forall m \in \mathbb{Z}_+,$$

then  $M_g(u) = 0$  in  $\tilde{\mathbb{C}}$ . □

We have compatibility of the generalized mean value with that of a distribution as stated in the following.

**Proposition 9.** *If  $T \in \mathcal{B}'_{ap}$ , then  $M_g(i_{ap}(T)) = M(T)$  in  $\mathbb{C}$ .*

*Proof.* We have

$$M_g(i_{ap}(T)) = \lim_{X \rightarrow +\infty} \frac{1}{X} \int_0^X i_{ap}(T)(h) dh = \left( \lim_{X \rightarrow +\infty} \frac{1}{X} \int_0^X (T * \rho_\varepsilon)(h) dh \right)_\varepsilon,$$

where  $\rho \in \Sigma$  and  $\rho_\varepsilon(\cdot) = \frac{1}{\varepsilon} \rho(\frac{\cdot}{\varepsilon})$ . Let  $\varphi \in \mathcal{D}$  and  $\int_{\mathbb{R}} \varphi(x) dx = 1$ , then

$$M(T) = \lim_{X \rightarrow +\infty} \frac{1}{X} \int_0^X (T * \varphi)(h) dh.$$

We have

$$M_g(i_{ap}(T)) - M(T) = M(T * (\rho_\varepsilon - \varphi)), \forall \varepsilon \in I.$$



In view of formula (VI.9.2) of [6], we obtain

$$M_g(i_{ap}(T)) - M(T) = M(T) \int_{\mathbb{R}} (\rho_\varepsilon(x) - \varphi(x)) dx,$$

as  $\forall \varepsilon \in I, \int_{\mathbb{R}} (\rho_\varepsilon(x) - \varphi(x)) dx = 0$ , then  $M_g(i_{ap}(T)) = M(T)$  in  $\mathbb{C}$ .  $\square$

**Remark 3.** We can introduce a new association within  $\mathcal{G}_{ap}$  with the aid of the generalized mean value  $M_g$ .

**Definition 10.** A generalized trigonometric polynomial is a generalized function  $[(P_\varepsilon)_\varepsilon]$ , where

$$P_\varepsilon(x) = \sum_{n=1}^l c_{\varepsilon,n} e^{i\lambda_{\varepsilon,n}x}, \quad (c_{\varepsilon,n})_\varepsilon \in \widetilde{\mathbb{C}} \quad \text{and} \quad (\lambda_{\varepsilon,n})_\varepsilon \in \widetilde{\mathbb{R}}, \quad n = 1, \dots, l.$$

**Proposition 10.** Every generalized trigonometric polynomial is an almost periodic generalized function.

*Proof.* Let  $P_\varepsilon(x) = \sum_{n=1}^l c_{\varepsilon,n} e^{i\lambda_{\varepsilon,n}x}$  where  $(c_{\varepsilon,n})_\varepsilon \in \widetilde{\mathbb{C}}$  and  $(\lambda_{\varepsilon,n})_\varepsilon \in \widetilde{\mathbb{R}}$ ,  $n = 1, \dots, l$ , we have

$$\forall \varepsilon \in I, \quad \sum_{n=1}^l c_{\varepsilon,n} e^{i\lambda_{\varepsilon,n}x} \in \mathcal{B}_{ap},$$

moreover  $\exists m \in \mathbb{Z}_+, \exists m' \in \mathbb{Z}_+, |\lambda_{\varepsilon,n}| = O(\varepsilon^{-m})$  and  $|c_{\varepsilon,n}| = O(\varepsilon^{-m'}), \varepsilon \rightarrow 0$ . Consequently  $\forall k \in \mathbb{Z}_+, \exists C > 0$  such that

$$\begin{aligned} |P_\varepsilon|_{k,\infty} &\leq \sum_{j \leq k} \sum_{n=1}^l |\lambda_{\varepsilon,n}|^j |c_{\varepsilon,n}| \\ &\leq \left( \sum_{j \leq k} |\lambda_{\varepsilon,1}|^j + |\lambda_{\varepsilon,2}|^j + \dots + |\lambda_{\varepsilon,l}|^j \right) c' \varepsilon^{-m'} \\ &\leq k (c \varepsilon^{-m} + c^2 \varepsilon^{-2m} + \dots + c^k \varepsilon^{-km}) c' \varepsilon^{-m'} \\ &\leq C \varepsilon^{-m''}, \quad \text{where } C = c' k l (c + c^2 + \dots + c^k), \quad m'' = km + m', \end{aligned}$$

this show that  $(P_\varepsilon)_\varepsilon \in \mathcal{M}_{ap}$ . In a similar way we show that if  $(\lambda_{\varepsilon,n})_\varepsilon \in \mathcal{N}[\mathbb{R}]$  and  $(c_{\varepsilon,n})_\varepsilon \in \mathcal{N}[\mathbb{C}]$ , then  $(P_\varepsilon(x))_\varepsilon \in \mathcal{N}_{ap}$ .  $\square$

Let  $u \in \mathcal{G}_{ap}$  and  $\tilde{\lambda} = [(\lambda_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}}$ , then  $ue^{-i\tilde{\lambda}x} = (u_\varepsilon e^{-i\lambda_\varepsilon x})_\varepsilon \in \mathcal{G}_{ap}$ , so the generalized mean value  $M_g(u e^{-i\tilde{\lambda}x})$  is a well defined element of

$\tilde{\mathbb{C}}$ . Define

$$a_{\tilde{\lambda}}(u) = M_g \left( u e^{-i\tilde{\lambda}x} \right),$$

and the generalized spectra of  $u$ ,

$$\Lambda_g(u) = \left\{ \tilde{\lambda} \in \tilde{\mathbb{R}} : a_{\tilde{\lambda}}(u) \neq 0 \text{ in } \tilde{\mathbb{C}} \right\}.$$

**Remark 4.** *If a function or a distribution  $u$  is almost periodic, then its spectra is at most countable, see [1] and [6].*

**Proposition 11.** *Let  $P = \left[ \left( \sum_{n=1}^l c_{\varepsilon,n} e^{i\lambda_{\varepsilon,n}x} \right)_{\varepsilon} \right]$  be a generalized trigonometric polynomial, then*

$$\Lambda_g(P) = \left\{ [(\lambda_{\varepsilon,n})_{\varepsilon}] : n = 1, \dots, l \right\}.$$

*Proof.* A direct computation gives the result.  $\square$

**Remark 5.** *We are indebted to the referee for the following example of an almost periodic generalized function having uncountable generalized spectra. Let  $u = [(u_{\varepsilon})_{\varepsilon}]$ ,  $u_{\varepsilon}(x) = 1 + e^{ix}$ ,  $\forall \varepsilon \in I$ , by proposition (11),  $\Lambda_g(u) = \{[(A_{\varepsilon})_{\varepsilon}] : A_{\varepsilon} = \{0, 1\}\}$ , as the set  $\{(\lambda_{\varepsilon})_{\varepsilon} : \lambda_{\varepsilon} = \{0, 1\}, \varepsilon \in I\} \subset \Lambda_g(u)$  is uncountable, then the generalized spectra of  $u$  is too.*

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